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W-Algebras and the Embedding of Toda Theories in WZNW Theories. *

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Abstract: It is shown that Polyakov's embedding of two-dimensional gravitational theory in a Kac-Moody algebra can be generalized to an embedding of any Toda field theory in a WZNW theory and that many of the properties of the Toda theory become much simpler and intuitive in the WZNW context. In particular, the appearance of Zamolodchikov W-algebras in the Toda theories is seen to be quite natural and their computation reduces to a relatively simple algorithm. In the quantized version of the theory there also appears an interesting formula connecting the Kac-Moody and Virasoro centres.

1 Introduction.

It is well-known that bosonic string theory in less than twenty-six dimensions induces a two-dimensional gravitational theory whose Lagrangian is just the Liouville one [1], and recently, in order to facilitate the quantization of this theory it was proposed by Polyakov [2] that the Liouville theory be embedded in a Kac-Moody (KM) theory. Working in the light-cone gauge, Polyakov actually carried out this program only for the left and right moving currents seperately, but it was later shown [3] that the embedding could be carried out for both sides simultaneously and that it could be generalized to the Toda field theories [4], which are the natural first generalization of the Liouville theory. In fact it was shown [3] that Toda theories could be regarded Wess-Zumino-Novikov-Witten (WZNW) theories

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(i) The W-algebras of Zamolodchikov [5], which appear rather mysteriously in the Toda theory are seen to have a very natural origin in the constrained WZNW theory, namely, as the algebras of local gauge-invariant polynomials in the currents

(ii) In a certain class of gauges these polynomials reduce to the currents themselves and in these gauges the W-algebras find another interpretation as the Dirac star algebra of the constrained currents. This interpretation is important from the practical point of view because it allows the W-algebras to be computed from the KM algebras in a relatively simpler manner[6].

(iii) One obtains a very simple and interesting formula connecting the centres of the WZNW Kac-Moody (KM) algebras and the Toda Virasoro algebras.

(iv) The procedure can be generalized [7] to obtain a whole family of conformally invariant integrable field theories that interpolate between the WZNW and Toda theories, and describe interacting WZNW fields belonging to subgroups of the original WZNW group.

(v) The general solutions of the new conformally invariant integrable systems can be obtained in a relatively simple manner from the well-known general solutions of the WZNW theories.

(vi) In all the new reduced conformally-invariant theories (including the Toda theories) a two-dimensional gravitational field emerges in a very natural way. Thus one obtains a (two-dimensional) unified theory of gravity and interacting WZNW fields.

(vii) For the generalized theories the formula connecting the KM and Virasoro centres changes only in that the parameters acquire a more general interpretation.

(viii) The procedure generalizes even further to yield non-conformally-invariant theories, and these include the affine Toda theories, for example the sinh-Gordon theory.

The present talk is actually the first part of two talks concerning these six points, the other being the one by Andreas Wipf. This talk will be concerned only with the reduction to Toda theories, and and will concentrate on the question of the W-algebras. The formula for connecting the centres will also be derived, but the derivation of the general solution and the role of the gravitational field will be left to the other talk as they are better described in the general case.

To refresh our memories and to fix the notation we first recall some relevant properties of two-dimensional conformal field theories, including the definition of W-algebras and the WZNW and Toda Lagrangians

2 Recall of Conformal Field Theory and W-Algebras.

We begin by recalling the situation for conformal invariance in more than two dimensions ($D > 2$). Let $L(\phi(x))$ be the Lagrangian density for any set of tensor fields $\phi(x)$ and $T_{\mu\nu}(\phi(x))$ the corresponding energy-momentum tensor density. If $L(\phi(x))$ is conformally invariant then according to Noether's theorem the generators $[L_{\mu\nu}, P_\mu, S_\mu, D]$ of the conformal group are moments of $T_{\mu\nu}$. For example, for the dilation D one has $D = \int x^\nu T_{\nu o} d^{D-1}x$. In all the $D > 2$ cases the conformal group is finite-dimensional ($(D+1)(D+2)/2$-dimensional actually), and thus involves only a finite number of moments of $T_{\mu\nu}$. It is also semi-simple and thus admits no central extensions.

In two dimensions the situation is different. If $x = (x_1, x_2)$ are the usual Cartesian coordinates, then the conformal group consists of all transformations of the form $z \to f(z)$ and $w \to \bar{f}(w)$, where $z, w = x_1 \pm x_2$, or $z, w = x_1 \pm x_2$, according as the signature is Minkowskian or Euclidean, and $f(z)$ and $g(w)$ are arbitrary analytic functions. Thus it is an infinite-dimensional group and is a direct product of a left and a right part. Furthermore, it is well-known that each part admits one central extension [1]. For conformally invariant Lagrangians the (three component) energy-momentum tensor density $T_{\mu\nu} = [T_{zw}, T_{zz}, T_{ww}]$ reduces to $[T_{zw} = 0, \quad T_{zz} = L(z) \quad \text{and} \quad T_{ww} = \bar{L}(w)]$, and the Noether generators of the conformal group consist of *all* the moments i.e. consist of the quantities

$$L_n = \oint z^{n-1} L(z) dz \quad \text{and} \quad \bar{L}_n = \oint w^{-1} n \bar{L}(w) dw, \qquad z, w \in Z. \tag{1}$$

From the structure of the conformal group it follows that the $L(z)$ and $\bar{L}(w)$ commute with each other and that each satisfies a Virasoro algebra i.e. an algebra of the form

$$[L(z), L(z')] = 2L(z)\partial_z \delta(z - z') + \partial_z L(z) \delta(z - z') + \frac{c}{12} \partial_z^3 \delta(z - z'), \tag{2}$$

where the last term is the central extension and c is a constant that depends on the original Lagrangian.

The tensors with respect to the conformal group are called *primary* fields and have the transformation properties

$$\phi(\mathbf{x}) \to \left(\frac{\partial z}{\partial z'}\right)^j \left(\frac{\partial w}{\partial w'}\right)^{\bar{j}} \phi(\mathbf{x}'), \tag{3}$$

where the quantities j and $\bar{j}$ are called conformal weights and are often integers.

With these properties recalled let us turn to the definition of W-algebras. According to Zamolochikov, who first introduced them [7], a W-algebra is an extension of a Virasoro algebra by primary fields, such that the Poisson bracket (or commutators) of any two primary fields is a polynomial in the fields and their derivatives (both primary and Virasoro), the order of the polynomial being less than the combined order of the two primary fields. In other words a W-algebra consists of the Virasoro algebra, the transformation law (3) (with one of the coordinates (w,say) dormant) and a set of Poisson bracket (or commutation relations) of the form

$$[\phi^{(a)}(z), \phi^{(b)}(z')] = P^{(a,b)}(\phi(z), L(z), \delta(z-z')), \tag{4}$$

where $P^{(a,b)}$ is polynomial of lower order than (a+b) in $L(z)$, $\phi(z)$, $\delta(z-z')$ and their derivatives. In counting the order the delta function and the derivative are each given unit weight.

3 WZNW and Toda Lagrangians.

Two standard examples of 2-D conformal field theories are the Wess- Zumino-Novikov-Witten (WZNW) theory and the Toda theory. The WZNW Lagrangian [8] takes the form

$$L_{WZ} = \frac{k}{2}\oint d^2x \operatorname{tr}(g^{-1}(x)\partial g(x))^2 + \frac{2k}{3}\oint d^3x \operatorname{tr}(g^{-1}(x)\partial g(x))^3, \tag{5}$$

where the three-dimensional integral is over a space whose boundary is the two-dimensional one of the first (kinetic) integral. As a result of the addition of the three-dimensional integral, whose variation is purely topological, the field equations of the theory take the form

$$\partial_w J(x) = 0 \quad \text{and} \quad \partial_z \bar{J}(x) = 0, \tag{6}$$

where

$$J(x) = g^{-1}(x)\partial_z g(x) \quad \text{and} \quad \bar{J}(x) = (\partial_w g(x))g^{-1}(x).$$

The field equations mean, of course, that the currents $J(x)$ and $\bar{J}(x)$ are functions only of z and w respectively and from the symmetry of L_{WZ} with respect to (rigid) left and right group multiplication ($g \to hg$ and $g \to gh$), and the Noether theorem, it follows that they satisfy Kac-Moody (KM) algebras with centres k. Thus $J(z)$, for example, satisfies the KM algebra

$$[J_a(z), J_b(z')] = f^c_{ab} J_c(z)\delta(z - z') + k\delta_{ab}\partial_z \delta(z - z'). \tag{7}$$

The Toda Lagrangian [4], on the other hand, takes the form

$$L_{Toda} = \int d^2x[C_{ij}\phi^i(x)\partial\phi^j(x) + \exp(K_{ij}\phi^i(x)\phi^j(x))], \tag{8}$$

where C and K are the Coxeter and Cartan matrices for any semi-simple simple Lie group. Thus to every Dynkin diagram there is associated a Toda field theory.

Recently it has been shown [9][10] that Toda field theories admit W-algebras, the W's being coefficients in an equation called the Gelfand Dickey equation [10][11]. This equation is a linear differential (or pseudo-differential) equation of the same order as the dimension of the defining representation of G, and which is satisfied by certain left-and right-moving functionals of the Toda fields. Its role in our discussion will be to help identify the W-algebras.

4 Lie Algebraic Technicalities.

Before proceeding to describe the WZNW → Toda reductions there are some group theoretical technicalities that are needed and will now be described.

First, the simple WZNW groups G which are used for the reduction will be the (maximally non-compact) ones generated by the *real* linear span of the Cartan generators i.e. by the generators $[H_i, E_\alpha]$ in conventional notation. For the A and D algebras, for example, these are the groups SL(n,r) and SO(n,n). Within the Cartan algebra there exists an element H such that each of the simple roots E_{α_i} is an eigenvector of H with eigenvalue unity.

$$[H, E_{\alpha_i}] = E_{\alpha_i} \qquad i = 1, 2...l, \tag{8}$$

where l is the rank. This can easily be seen by noting that H = s.**H** where s is the sum over the fundamental coweights (=half the sum of the positive coroots) and thus the inner product of s with each simple root is unity. Note that H then provides an integer grading of the whole Lie algebra,

$$[H, E_\alpha] = hE_\alpha \quad \text{where} \quad h \in Z, \tag{9}$$

and that G admits a local Gauss decomposition $G = ABC$, where B is the Cartan group and A and C are the (nilpotent) groups generated by the root vectors E_α for negative and positive α respectively. (This decomposition may not be global, but the parameter space may be divided into a finite number of patches on each of which the decomposition is valid up to left- or right-multiplication with a constant group element).

At the KM level we have, correspondingly,

$$[H(z), J^B(z')] = 0 \quad \text{except} \quad [H(z), H(z')] = k\partial_z \delta(z - z')\mathrm{tr}H^2, \tag{10}$$

and

$$[H(z), J_h^\alpha(z')] = hJ_h^\alpha(z)\delta(z - z'). \tag{11}$$

5 Conformal-Invariant Reduction.

We come now to the central point, which is the reduction of the WZNW theories to Toda theories. The reduction is effected by setting some components of the WZNW currents equal to non-zero constants. However, since the current components are conformal vectors i.e. have conformal weights (1,0) and (0,1) with respect to the conformal group generated by the components $L(z)$ and $\bar{L}(w)$ of the WZNW energy-momentum tensor, the produre of setting some of them equal to constants would appear to violate conformal invariance. The way out of this difficulty is to note that the generators of the conformal group are not unique but are part of a two-parameter family, and to choose instead members of the family with respect to which the required components of the currents are conformal scalars. The required members of the family are obtained by replacing $L(z)$ and $\bar{L}(w)$ by

$$\Lambda(z) = L(z) + \partial_z H(z), \quad \text{and} \quad \bar{\Lambda}(w) = \bar{L}(w) - \partial_w H(w). \tag{12}$$

It is to be noted that $\Lambda(z)$ and $\bar{\Lambda}(w)$ are again Virasoro operators i.e. satisfy Virasoro algebras of the form (2). The only difference is that the centres c change to $c - 12k\mathrm{tr}H^2$. It will turn out that $\Lambda(z)$ and $\bar{\Lambda}(w)$ are actually the components of the improved (i.e.traceless) energy-momentum tensors of the Toda theories.

Once the crucial change (12) has been made the rest is almost automatic. With respect to the conformal group generated by $\Lambda(z)$ the KM currents $J(z)$ are no longer vectors of conformal weight (1,0) but have the following transformation properties:

(i) Except for $H(z)$ the currents $J^B(z)$ belonging to the Cartan subgroup B are still vectors i.e. have conformal weights (1,0).

(ii) the current $J^H(z) \equiv H.J(z)$ now transforms not as a spin-one vector but as a spin-one *connection*.

(iii) The currents $J^{\alpha}(z)$ transform as conformal tensors (primary fields) of conformal weight $(1+h)$.

Thus, in particular, *the currents $J(z)$ of grade $h=-1$ transform as conformal scalars.* Similiarly the currents $\bar{J}(w)$ of grade $h=1$ transform as conformal scalars.

With this information in hand we are ready to impose the constraints, namely,

$$J^{\alpha}_{-1}(z)=J^{\alpha}_{-1}(0)\neq 0, \quad \text{and} \quad J^{\alpha}_{h}(z)=0, \quad h<-1, \tag{13}$$

and similiarly for the right-handed currents $\bar{J}(w)$ with $h>1$. In (13) the set of constraints with non-zero right-hand-side do not break the conformal invariance generated by the new Virasoro operator $\Lambda(z)$ since they are scalars with respect to this operator, and the set of constraints with zero right-hand-side are added so that the complete system of constraints is first class. In order to obtain an intuitive feeling for the meaning of the constraints (13) let us consider the case of G=SL(n,R), in which case the constrained current $J(z)$ takes the form

$$J_{constr.}(z)=\begin{pmatrix} J_{11}(z) & J_{12}(z) & J_{13}(z) & \ldots\ldots\ldots & J_{1n}(z) \\ J_{21}(0) & J_{22}(z) & J_{23}(z) & \ldots\ldots\ldots & J_{2n}(z) \\ 0 & J_{23}(0) & J_{33}(z) & \ldots\ldots\ldots & J_{3n}(z) \\ 0 & 0 & J_{34}(0) & \ldots\ldots\ldots & J_{4n}(z) \\ 0 & 0 & 0 & \ldots\ldots\ldots & J_{5n}(z) \\ \ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots & \ldots\ldots \\ 0 & 0 & 0 & J_{nn-1}(0) & J_{nn}(z) \end{pmatrix}$$

Note that the constraints can also be expressed as

$$J^{\alpha}=M \quad \text{and} \quad \bar{J}^{\alpha}=N, \quad \text{for} \quad \alpha>0, \tag{14}$$

where M and N are constants matrices of grade minus one and plus one respectively.

The constraints (13) are not invariant with respect to general KM transformations, $J(z)\rightarrow U(z)^{-1}J(z)U(z)+U(z)^{-1}\partial_z U(z)$ but there exists a residual group of KM transformations with respect to which they are invariant. These are the KM transformations for which $U(z)$ lies in the group A of the Gauss decomposition which is generated by the negative root vectors (E^{α}, for $\alpha<0$). Thus they are just the transformations that would be generated by the constraints themselves. The idea will be to regard these residual KM transformations as gauge transformations and regard only those functions, or functionals, of $J(z)$ which are invariant with respect to this gauge group as physical. Thus finally we have (r-l)/2 constraints and (r-l)2 gauge degrees of freedom, where $r=\dim G$, leaving just l physical fields altogether. It is possible to choose the gauge (at least locally) so that the physical fields are just the currents $J^{B}(z)$ belonging to the Cartan subgroup B.

6 Field Equations.

It is easy to see that the constraints (13) are consistent with the WZNW field equations (6), indeed are special solutions of some of them, and hence the WZNW field equations can be reduced to field equations for the unconstrained components of the current $J(z)$. After some simple algebra one finds that the reduced field equations take the following form

$$\partial_{\bar{z}} J^B(x) = [b(x)Mb^{-1}(x), N], \tag{15}$$

and

$$J^A(x) \equiv a^{-1}(x)\partial_z a(x) = b(x)Mb^{-1}(x), \qquad \bar{J}^C(x) \equiv (\partial_w c(x))c(x)^{-1}. \tag{16}$$

where M and N denote the constant matrices defined in (14).

The most interesting feature of these equations is that the equation for $J^B(z)$ does not involve the fields $J_h(z)$ for $h \neq 0$ and thus are self contained. Furthermore, it is easy to verify that they can be derived from the effective Lagrangian

$$L_{\text{eff}}(b(x)) = L_{WZNW}(b(x)) \quad + \quad \int \text{tr}(b(x)Mb^{-1}(x)N), \quad \text{where} \quad b(x) \in B. \tag{16}$$

which, since B is abelian and M and N have weights ± 1, can be written as

$$L_{\text{eff}}(b(x)) = \oint (\partial\phi(x))^2 \quad + \quad \oint trDe^{2\phi(x)}, \tag{17}$$

where $b(x) = e^{\phi(x)}$ and D is the diagonal matrix MN. But the Lagrangian (17) is just the Toda Lagrangian and thus the reduction (14) has converted the WZNW theory into a Toda theory, as required. As mentioned earlier, the general solutions of of the Toda field equations can be obtained from the solutions of the WZNW equations. However the derivation will be given not here but in the second talk, as it is special case of the derivation for the general conformal reduction.

7 Identification of the W-Algebras.

In this section we show that the W-algebras that appear in Toda theories become much more understandable and tractable when the Toda theories are regarded as reduced WZNW theories. First we identify them by means of the equation $\partial_z g(z) = J(z)G(z)$ connecting the WZNW fields $g(z)$ with their currents $J(z)$, which can be regarded as first-order differential equations for $g(z)$, given $J(z)$. Since the matrices $J(z)$ act only on the columns of the matrices $g(z)$ these equations can be regarded as a set of linear equations of the form

$$\partial_z \psi(z) = J(z)\psi(z), \tag{18}$$

for the columns $\psi(z)$ of $g(z)$. Now, with respect to the gauge- group $U(z)$ mentioned earlier, the components of the vectors $\psi(z)$ transform as vectors, and for every simple Lie group G there is just one component, which we shall call $\psi_o(z)$ whichr transforms as a scalar. It is then natural to eliminate the other components of $\psi(z)$ from (18) and obtain a higher-order, but gauge-invariant, equation for $\psi_o(z)$. For the A,B,C groups this equation will be a liear differential equation of the form

$$[\partial^n + c_2(J)\partial^{n-2} + c_3(J)\partial^{n-3} \ldots\ldots + c_{n-1}\partial + c_n(J)]\psi(z) = 0, \tag{19}$$

where $\partial \equiv \partial_z$ and the $c(J)$'s are polynomials in the currents $J(z)$ and their derivates. For the other groups it is a similiar equation, but is pseudo-differential in the sense that it contains a few negative powers of ∂ as well. The interesting point is that these equations are just the Gelfand-Dicke equations mentioned earlier and thus the coefficients $c(J)$ are just the elements of a bases of the W-algebras of Toda theory. On the other hand from the above derivation of the Gelfand-Dicke equation from the WZNW theory one finds that not only are the $c(J)$ gauge-invariant by construction, but they are polynomials in the constrained currents and their derivatives. Thus we immediately obtain an identification of W-algebras as algebras of local gauge-invariant polynomials of the constrained currents. Furthermore, since it can be shown that the $c(J)$ form a complete basis for all the local gauge-invariant polynomials in the currents, one obtains an identication of the W-algebras as *the* algebras of local gauge-invariant polynomials of the currents.Note that this identification, which is very natural in the constrained WZNW theory, does not exist within the Toda context. It becomes apparent only when the Toda theory is embedded in the WZNW theory.

8. Choices of Gauge.

Although the identification of the W-algebra of Toda theory as the algebra of local gauge-invariant polynomials of the constrained WZNW theory is very natural and intuitive, and is, of course, by definition, gauge-invariant, the various properties of the algebras only become evident in particular gauiges. Among the gauges which are useful for exrtracting the properties are the diagonal gauge and the Drinfeld-Sokolov (DS) gauges [12]. The diagonal gauge is the one in which all the off-diagonal current components (except the constant components of grade ± 1) are set equal to zero. It does not represent a complete gauge-fixing but it gives the free-field representation of the W-algebras and it allows the base elements to be put in one-one correspondence with the KM Casimirs. In particular it shows that the orders of the gauge-invariant polynomials are just the orders of the Casimirs of the original WZNW group G. The DS gauges are a set of gauges which constitute a complete gauge fixing and have the property the local gauge-invariant polynomials reduce to the current components themselves. They can be characterized by means of the orders of the Casimirs and by the subgroup $SL(2,R)$ of G described in section 11 as follows: Let h_i for $i = 1,2...l$ denote the orders of the Casimirs. Then put all the components of the constrained currents (apart from the constant components of grade ± 1) equal to zero except one component of each grade $h_i - 1$ (subject only to the condition that each of the components chosen belongs to a different irreducible representation of the $SL(2,R)$ subgroup). For example, for the $SL(n,r)$ constrained currents depicted earlier one takes just one component in each of the upper-triangular rows which are slanted parallel to the diagonal (subject only to the condition that they cannot be generated from one another by the step operators of $SL(2,R)$), in particular one can take the elements $J_{1n}(z)$ for $n > 0$ in the top row of the matrix $J^{contr.}(z)$ depicted earlier. The DS gauges are in a certain sense the opposite of the diagonal gauges in that the l non-vanishing (z-dependent) components of the constrained currents $J(z)$ are all off-diagonal. In fact they are as far off-diagonal as the orders of the Casimirs specify. The reason that the polynomials reduce to linear functions of the currents in in the DS gauges can be seen intuitively by noting that if one writes $J = M + j$,where M is the grade minus-one constant matrix discussed earlier and j is a matrix constructed according to the DS specifications, then the leading term in the Casimir $\mathrm{tr}(M+j)^{h_i}$ will be of the form $mj_{h_i-1} + P(j)$, where m is a constant and P(j) is a polynomial in current components of lower order.

9. Interpretation of W-agebras as Dirac Star Algebras.

The direct method to compute the W-algebras is to construct a basis for the local gauge-invariant polynomials explicitly and then compute their Poisson, or commutator, brackets. However, this method is extremely complicated and has been carried out only for the lowest-dimensional groups. A much simpler method is provided by the DS gauges, in which, as we have seen, we have $P_{h_i}(J) = j^{DS}_{h_i-1}$, where the P_{h_i} are a basis for the W-algebra. In these gauges, the currents are constrained by the original constraints and the gauge-fixing and the total set of constraints form a second-class system. Accordingly, in the DS gauges the W-algebra reduces to the Dirac star algebras for the DS currents,

$$[P_{h-i}, P_{h_k}] = [J^{DS}_{h_i-1}, J^{DS}_{h_k-1}]^* = [J^{DS}_{h_i-1}, J^{DS}_{h_k-1}] - [J^{DS}_{h_i-1}, C_\alpha][C_\alpha, C_\beta]^{-1}[C_\beta, J^{DS}_{h_k-1}]. \tag{20}$$

Furthermore since the constraints are all *linear* in the currents the C_α can be replaced by the components J_α of the currents which are set equal to zero in the DS prescription. Thus, finally,

$$[P_{h_i}, P_{h_k}] = [J^{DS}_{h_i-1}, J^{DS}_{h_k-1}]^* = [J^{DS}_{h_i-1}, J^{DS}_{h_k-1}] - [J^{DS}_{h_i-1}, J_\alpha][J_\alpha, J_\beta]^{-1}[J_\beta, J^{DS}_{h_k-1}], \tag{21}$$

where $J_\alpha \neq J^{DS}$. Thus not only do we get an identification of the W-algebra as a Dirac star algebra of the original KM algebra but we find that the right hand side is expressible completely in terms of the original KM brackets (Poisson or commutator).

10. Computation of W-algebras

The formula (21) provides us with a method of computing the W-algebras directly from the KM algebras without actually computing the base-elements of the W-algebras explicitly, and, in principle we should now commence to compute using this formula. However, even this computation is not so easy. Luckily there is a trick by which it can be simplified still further. This is to regard the algebra as implementing the infinitesimal changes in the currents induced by W-transformations,

$$\delta_{h_i} J(z) = \oint dy a_{h-i}(y)[J_{h_i-1}, J(z)]^* \tag{22}$$

where the a(y) are arbitrary parameters. Using the definition of the Dirac star bracket this can be written as

$$\delta_{h_i} J(z) = \oint dy(a_{h_i}[J_{h_i-1}, J(z)] + a_\alpha(y)[J_\alpha, J(z)]), \tag{23}$$

where

$$a_{h_i}[j_{h_i-1}, j_\alpha] + a_\beta[j_\beta, j_\alpha] = 0. \tag{24}$$

In principle we should now solve (24) for the $a_\beta(y)$, substitute into (23) and compute the change induced by the W's. But since (22) is an implementation of the infinitesimal W-transformations and these are just the constrained KM transformations, there is actually a simpler way to compute its right-hand-side. This is to use the converse of the Noether theorem and replace (23) for *any* KM variation i.e. any $a_i(y)$ and $a_\alpha(y)$ by the corresponding KM matrix KM transformation

$$(\delta_{h_i} J(z)) = [M(z), J^{DS}(z)] + M'(z), \tag{25}$$

and then note that the W-transformations are the subset of these transformations which preserve the DS form of the current i.e. such that $\delta J(z)$ is a DS current of the same form as $J^{DS}(z)$. In other words the W-transformations correspond exactly to those $M(z)$'s in (23) which are such that the variation $\delta J(z)$ has the same DS form as the current $J(z)$ itself. Imposing this condition on $M(z)$ is equivalent to solving (24) for the a_β and leaves just l components of $M(z)$ free, corresponding to the l parametrs $a_{h_i}(y)$. However, being just a matrix equation, (25) is much easier to solve than the system (23)(24), and, in fact, it turns out that, because of the nilpotency of the gauge-group it can be solved in an iterative, polynomial manner. Thus finally all we have to do is solve the equation (25) in this iterative polynomial manner and read off the W-algebra from the result. This we have done [6] for the groups A_2, B_2 and G_2. The result for G_2 is particularly interesting because the W-algebra involves the bracket $[P_6(J(z)), P_6(J(z'))]$ of two sixth-order polynomials (corresponding to the sixth-order Casimir of G_2), and so is quite difficult to compute by direct means. In fact, as far as we know, it has not yet been computed directly.

13. The Subgroup SL(2,R).

We conclude by considering a property of the Toda reduction that is not shared by the more general reductions to be considered in the second talk, namely, the existence of an $SL(2,R)$ subgroup of the WZNW group G which has very special properties. This is the $SL(2,R)$ subgroup generated by the matrices H, M and M^*, where H and M are the matrices discussed earlier and M^* is a third element determined by the first two, and whose explicit form is not needed. This group can be used to define the DS gauges in a simple linear manner and governs many of their properties. For example, the DS gauge in which current components are highest weights with respect to this subgroup, is one in which the base elements of the W-algebra are primary fields (which establishes that these algebras are indeed W-algebras in the sense of Zamolodchikov) and the question of the pseudo-differentiability of the Gelfand-Dicke equation can be reduced to the simple algebraic question as to whether the defining representation of G is irreducible with respect to the $SL(2, R)$ subgroup. From this it is easy to see that it is not pseudo-differential iff $G = A, B, C$ or G_2.

15. Formula for Centres.

Finally, as a first step towards the quantization of the theory we derive a formula connecting the KM and Virasoro centres which is valid for any highest weight (Fock) representation of the algebra, and should therefore be valid for any quantized version. The range of each centre seperately is not determined, but presumably other physical conditions for the quantization, such as unitarity, will restrict these ranges. The starting point is the well-known formula [8]

$$c = \frac{\dim G}{1 + g/k}, \tag{26}$$

where g is the Coxeter number of G which connects the KM and Virasoro centres for unconstrained KM theories. When the Virasoro operator is modified according to (12) this formula acquires an extra term from the $\partial_Z H(z)$ part and becomes

$$c = \frac{\dim G}{1 + g/k} - 12k\mathrm{tr}H^2. \tag{27}$$

What we now have to add is the contribution from the BRST ghosts due to the constraints. For this we observe that the current components to be constrained are primary fields with weights h_α for negative α. We must introduce a ghost pair for each of these constraints and from the usual formula for the contribution of primary fields of these weights for highest weight representations we see that the ghost contribution is just

$$c_{ghost} = \sum_{\alpha>0} [12h_\alpha(h_\alpha - 1) - 2]. \tag{28}$$

Using the Lie algebraic formulae

$$\sum_{\alpha>0} h_\alpha = 2\bar{\mathrm{s}}.\mathrm{s} \quad \text{and} \quad \sum_{\alpha>0} h_\alpha^2 = \frac{1}{2}\mathrm{Tr}H^2, = g\mathrm{tr}H^2 \tag{29}$$

where $\bar{\mathrm{s}}$ denotes half the sum of the positive roots, and Tr denotes trace in the adjoint representation, we see that the ghost contribution to the centre can be written as

$$c_{ghost} = l - dimG - 12g\mathrm{tr}H^2 + 24\mathrm{s}.\bar{\mathrm{s}}. \tag{30}$$

By adding the ghost contribution to (27) and using the Freudenthal-de Vries formula $gD = 12\mathrm{s}^2$ one obtains finally

$$c = l - 12(\surd(k+g)\bar{\mathrm{s}} - (\surd(k+g))^{-1}\mathrm{s})^2, \tag{31}$$

as the required expression for the centre. This expression resembles the expressions in rational conformal field theories, but it is to be noted that what is being squared in (31) is a vector, not a number. For simply-laced groups, for which $s = \bar{s}$, equation (31) simplifies to

$$c = l - \frac{gD}{k+g}(k+g-1)^2 = l[1 - g(g+1)\frac{(r-s)^2}{rs}], \tag{32}$$

where $s/r = k+g$. The last expression resembles the formula in rational conformal field theory even more closely, but it must be remembered that, without further information, the quantity r/s cannot be assumed to be rational.

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